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Algebraization of Spectral Problems in the Bargmann–Fock Representation

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Examples of quasi-exactly solvable hamiltonians in the Bargmann–Fock representation are given. The existence of an invariant subspace is studied as a result of hidden symmetry of the spectral problem.

KEY WORDS: quasi-exactly solvable hamiltonians; Bargmann–Fock space; algebraization.

1. QUASI-EXACTLY SOLVABLE DIFFERENTIAL OPERATORS

The most important problems in quantum mechanics are these dealing with spectral problems of differential operators. The eigenvalues of an observable can be verified in various experiments serving as a confirmation for theoretical predictions. The point-spectra of Hamilton operators are of prime interest since they correspond to energies of the considered systems. Since the most typical hamiltonians are the sum of laplacian and the scalar function (potential), the eigenproblem is a differential equation. A historical development of group-theoretical methods allowed to find an algebraic solution to some of the problems. As an example let us mention the harmonic oscillator that admits algebraic solution due to the Heisenberg–Weyl symmetry. The *algebraization* of the spectral problem is based upon the existence of a finite dimensional invariant subspace of the space of the original problem. In such a subspace there is a matrix representation of the hamiltonian allowing for an exact solution by means of algebraic methods.

The operator A will be called *quasi-exactly solvable* (Shifman, 1989; Turbiner, 1988; Ushveridze, 1994) if there is a point-spectrum preserving transformation G such that $A_G := G^{-1}\, AG$ is a combination of the generators of the

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Lie algebra $sl(2)$:

$$T^+ := 2jz - z^2 \frac{d}{dz}, \quad T^0 := -j + z\frac{d}{dz}, \quad T^- := \frac{d}{dz}, \tag{1}$$

with integral or semiintegral j.

The space P spanned by the basis $\{z^k\}_{k=0}^{2j}$, $P = lin\{z^k\}_{k=0}^{2j}$, is an invariant subspace of any combination of T^i and thus $G^{-1}\,AG$ admits algebraization. The property of quasi-exactly solvability indicates *hidden symmetry* of the system.

2. HARMONIC OSCILLATOR

In this section we present two examples of the algebraization procedure applied to harmonic oscillator and displaced harmonic oscillator. Both problems will be analyzed in the Bargmann–Fock representation (Schweber, 1967) of the quantum system in the space of the entire complex functions. The scalar product of the states in this representation reads

$$\langle\psi_1|\psi_2\rangle := \int \exp(-|z|^2)\bar{\psi}_1(z)\,\psi_2(z)d\mu(z),$$

with $d\mu(z) := \frac{1}{pi}\,dx\,dy$ for $z = x + iy$. In this representation there are natural candidates for the bosonic creation and annihilation operators:

$$a \leftrightarrow \frac{d}{dz}, \quad a^\dagger \leftrightarrow z, \tag{2}$$

since they preserve the Heisenberg–Weyl commutation relations $[a, a^\dagger] = [d/dz, z] = 1$. In the Bargmann–Fock representation the harmonic oscillator hamiltonian $H = a^\dagger a$ reads

$$H = z\frac{d}{dz} \tag{3}$$

and is quasi-exactly solvable for $G = Id$, i.e.

$$H_G := G^{-1}HG = T^0 + j. \tag{4}$$

There is no limitation for j and the harmonic oscillator eigenproblem admits algebraization in the subspace of arbitrary finite dimension—like in the case of the well known textbook solution.

The displaced harmonic oscillator $H = a^\dagger a + \lambda(a + a^\dagger)$ can be represented as

$$H = z\frac{d}{dz} + \lambda\left(z + \frac{d}{dz}\right). \tag{5}$$

Instead of a simple redefinition of operators (Schweber, 1967) we perform the transformation $G = \exp(f(z))$ (with an arbitrary analitic $f(z)$) and obtain

$$H_G = z\,\frac{d}{dz} + zA(z) + \lambda z + \lambda\,\frac{d}{dz} + \lambda A(z), \tag{6}$$

where $A(z) := f'(z)$. The operator H_G is quasi-exactly solvable if the condition

$$zA(z) + \lambda z + \lambda A(z) = C = \text{const.} \tag{7}$$

is satisfied. Choosing the constant $C = -\lambda^2$ and neglecting the additive constants we get

$$H_G = (T^0 + j) + \lambda T^-. \tag{8}$$

The comparison of matrix representations shows that point-spectra of (8) and (4) are up to a constant equal.

3. OSCILLATOR WITH POLYNOMIAL NONLINEARITY

There are several problems in studying optical properties of nonlinear materials where one considers harmonic oscillator hamiltonians perturbed by nonlinear terms (Bishop and Vourdas, 1987; Peřina *et al.*, 1994; Vourdas, 1992). Let us consider

$$H = a^\dagger a + \overline{\lambda} a^{\dagger 2} + \lambda a^2 \tag{9}$$

or, in the Bargmann–Fock representation

$$H = z\,\frac{d}{dz} + \lambda\,\frac{d^2}{dz^2} + \overline{\lambda} z^2. \tag{10}$$

The hamiltonian (9) possesses the $SU(1, 1)$ symmetry (Perelomov, 1986) and is considered in connection with squeezed states (Bishop and Vourdas, 1987; Vourdas, 1992). The $SU(1, 1)$ group is noncompact and thus possesses no finite dimensional representations. The hamiltonian H is not diagonal in the $SU(1, 1)$ coherent states representation due to the nontrivial influence of the raising and lowering operators. Thus the $SU(1, 1)$ symmetry does not result in the algebraization. We show that apart of the $SU(1, 1)$ symmetry the system possesses the $SU(2)$ symmetry. Transformation $G = \exp(f(z))$ of the hamiltonian (10) gives

$$H_G = z\,\frac{d}{dz} + \lambda\,\frac{d^2}{dz^2} + 2\lambda A(z)\,\frac{d}{dz} + \lambda A'(z) + zA(z) + \overline{\lambda} z^2 + \lambda A^2(z). \tag{11}$$

For $f'(z) = A(z) = \omega z$ the operator (11) is quasi-exactly solvable if $\omega^2\lambda + \omega + \overline{\lambda} = 0$ and yields

$$H_G = (1 + 2\omega\lambda)(T^0 + j) + \lambda T^- T^-. \tag{12}$$

Here again j is arbitrary and one can find arbitrary many eigenvalues and eigenstates.

As an example we calculate explicitly three eigenstates of (11). Fixing $j = 1$ we diagonalize H_G in the space spanned by $\{1, z, z^2\}$ and get

$$E_1 = 1 + 2\omega\lambda, \quad \psi_1(z) = N_1 G(z),$$
$$E_2 = 2 + 4\omega\lambda, \quad \psi_2(z) = N_2 z G(z),$$
$$E_3 = 3 + 6\omega\lambda, \quad \psi_3(z) = N_2 z^2 G(z).$$

The normalization constant N_i is calculated with respect to the scalar product in the Bargmann–Fock space. The term $T^- T^-$ in (12) does not modify the spectrum and the transformation G has led us to the harmonic oscillator with modified frequency. If one considers the hamiltonian (9) in the first quantized form one realizes that it is indeed the case if λ is real. In that sense the algebraization presented above allows for slightly more general results.

4. TWO COUPLED OSCILLATORS

Quantum systems composed of interacting bosonic fields play an important role in quantum optics (Vourdas, 1992). Unfortunately the higher dimensional spectral problems are essentially more complex for algebraization (Shifman, 1989). There are only three groups admitting the differential realization in the space of complex polynomials of two variables namely $SU(2) \times SU(2)$, $SO(3)$, and $SU(3)$ (Shifman, 1989). We focus on the first one since it provides a natural representation of the harmonic oscillators. In this section we discuss a simple application of the product representation $sl(2) \times sl(2)$ spanned by the two sets of the generators $\{T^{0,\pm}\}$ and $\{\tilde{T}^{0,\pm}\}$ for algebraization of the spectrum of two coupled oscillators. Let a and b be bosonic operators satisfying $[a, b] = 0$. We consider the following hamiltonian with the $SU(1, 1)$ symmetry (Vourdas, 1992)

$$H = a^\dagger a + \omega b^\dagger b + \lambda a^\dagger b^\dagger + \overline{\lambda} ba, \tag{13}$$

which in the product of Bargmann–Fock representations reads

$$H = z\frac{\partial}{\partial z} + \omega x \frac{\partial}{\partial x} + \lambda z x + \overline{\lambda}\frac{\partial}{\partial z}\frac{\partial}{\partial x}. \tag{14}$$

We show that there is the $SU(2) \times SU(2)$ symmetry hidden behind (14). Since the problem is essentially two dimensional we seek for a transformation $G = \exp(f(z, x))$ which applied to (14) yields

$$H_G = z\frac{\partial}{\partial z} + zA_z(z, x) + \omega x \frac{\partial}{\partial x} + \omega x A_x(z, x) + \lambda z x$$

$$+\overline{\lambda}\left(A_x(z,x)A_y(z,x)+A_{xz}(z,x)+A_x(z,x)\frac{\partial}{\partial z}+A_z(z,x)\frac{\partial}{\partial x}+\frac{\partial}{\partial z}\frac{\partial}{\partial x}\right), \tag{15}$$

where $A_x := \frac{\partial}{\partial x} f(z,x)$, $A_z := \frac{\partial}{\partial z} f(z,x)$, and $A_{xz} := \frac{\partial^2}{\partial x \partial z} f(z,x)$. The choice $f(z,x) = \beta zx$ with β satisfying the condition $\beta^2\lambda + \beta(\omega+1) + \lambda = 0$ leads to the algebraization of (15) in the product representation

$$H_G = (1+\overline{\lambda}\beta)(T^0 + j_1) + (\omega + \overline{\lambda}\beta)(\tilde{T}^0 + j_2) + \overline{\lambda}T^-\tilde{T}^-. \tag{16}$$

The constants j_1 and j_2 determine the dimension of the product representation $P = lin\{z^k\}_{k=0}^{2j_1} \times lin\{x^k\}_{k=0}^{2j_2}$. They are arbitrary providing algebraization in arbitrary finite dimensions, e.g., for $j_1 = j_2 = 1/2$ one gets the matrix representation in the four dimensional space spanned by $\{1, x, z, xz\}$. Here again the system effectively decomposes into two harmonic oscillators with modified frequencies.

5. CONCLUSIONS

We presented an algebraic solution for several spectral problems of the oscillatory like hamiltonians for one- and two-mode bosonic fields. The natural representation of bosonic operators in the Bargmann–Fock space significantly simplifies the procedure. The algebraic solution was given not only for the harmonic oscillator in this representation but also for nonlinear and higher dimensional models. In each case considered in the paper algebraization can be performed in the invariant subspace of arbitrary finite dimension. It allows to find every finite number of eigenvalues. The hamiltonians studied in this paper find their application in quantum optics, e.g., in the information transmission (Vourdas, 1992).

With each of the models considered in the paper we may associate a family of operators that, being not quasi-exactly solvable in the sense of a hidden symmetry, are a small perturbation of the model admitting algebraization. The point spectrum of these operators can be obtained (if exists) via perturbation theory from the spectrum of the quasi-exactly solvable "parent." It justifies studying even the most artificial quasi-exactly solvable toy models.

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